Applications of Quadric Surfaces

Examples of quadric surfaces can be found in the world around us. In fact, the world itself is a good example. Although the earth is commonly modeled as a sphere, a more accurate model is an ellipsoid because the earth's rotation has caused a flattening at the poles. (See Exercise 49.)

Circular paraboloids, obtained by rotating a parabola about its axis, are used to collect and reflect light, sound, and radio and television signals. In a radio telescope, for instance, signals from distant stars that strike the bowl are all reflected to the receiver at the focus and are therefore amplified. (The idea is explained in Problem 18 on page 202.) The same principle applies to microphones and satellite dishes in the shape of paraboloids.

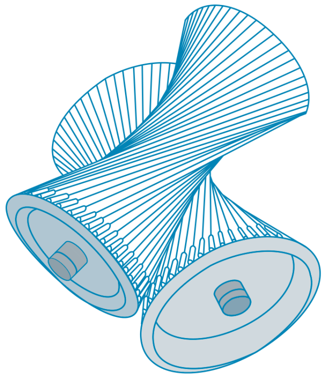
Cooling towers for nuclear reactors are usually designed in the shape of hyperboloids of one sheet for reasons of structural stability. Pairs of hyperboloids are used to transmit rotational motion between skew axes. (The cogs of the gears are the generating lines of the hyperboloids. See Exercise 51.)


David Frazier / Spirit / Corbis

A satellite dish reflects signals to the focus of a paraboloid.


Mark C. Burnett / Science Source

Nuclear reactors have cooling towers in the shape of hyperboloids.



Hperboloids produce gear transmission.

12.6 EXERCISES

1. (a) What does the equation $y = x^2$ represent as a curve in $\mathbb{R}^2$?
(b) What does it represent as a surface in $\mathbb{R}^3$?
(c) What does the equation $z = y^2$ represent?

2. (a) Sketch the graph of $y = e^x$ as a curve in $\mathbb{R}^2$.
(b) Sketch the graph of $y = e^x$ as a surface in $\mathbb{R}^3$.
(c) Describe and sketch the surface $z = e^y$.

3–8 Describe and sketch the surface.

3. $x^2 + z^2 = 1$ **4.** $4x^2 + y^2 = 4$

5. $z = 1 - y^2$ **6.** $y = z^2$

7. $xy = 1$ **8.** $z = \sin y$

9. (a) Find and identify the traces of the quadric surface $x^2 + y^2 - z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of one sheet in Table 1.
(b) If we change the equation in part (a) to $x^2 - y^2 + z^2 = 1$, how is the graph affected?
(c) What if we change the equation in part (a) to $x^2 + y^2 + 2y - z^2 = 0$?

10. (a) Find and identify the traces of the quadric surface $-x^2 - y^2 + z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of two sheets in Table 1.

(b) If the equation in part (a) is changed to $x^2 - y^2 - z^2 = 1$, what happens to the graph? Sketch the new graph.

11–20 Use traces to sketch and identify the surface.

11. $x = y^2 + 4z^2$

12. $4x^2 + 9y^2 + 9z^2 = 36$

13. $x^2 = 4y^2 + z^2$

14. $z^2 - 4x^2 - y^2 = 4$

15. $9y^2 + 4z^2 = x^2 + 36$

16. $3x^2 + y + 3z^2 = 0$

17. $\dfrac{x^2}{9} + \dfrac{y^2}{25} + \dfrac{z^2}{4} = 1$

18. $3x^2 - y^2 + 3z^2 = 0$

19. $y = z^2 - x^2$

20. $x = y^2 - z^2$

21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choice.

21. $x^2 + 4y^2 + 9z^2 = 1$

22. $9x^2 + 4y^2 + z^2 = 1$

23. $x^2 - y^2 + z^2 = 1$

24. $-x^2 + y^2 - z^2 = 1$

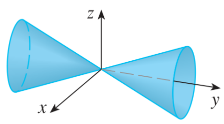
25. $y = 2x^2 + z^2$

26. $y^2 = x^2 + 2z^2$

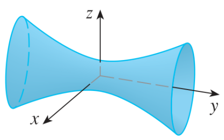
27. $x^2 + 2z^2 = 1$

28. $y = x^2 - z^2$

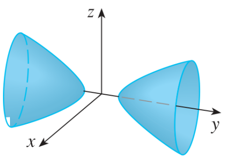
I



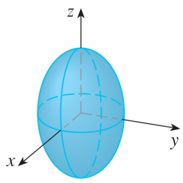
II



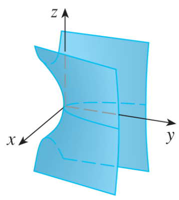
III



IV



V



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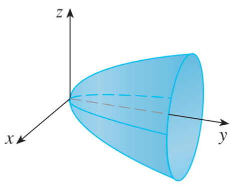
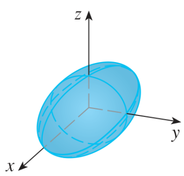
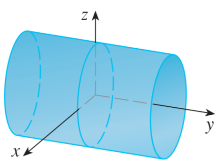


VII

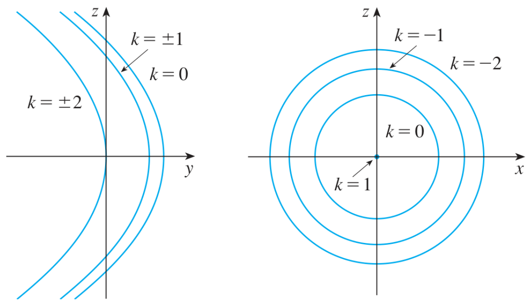


VIII

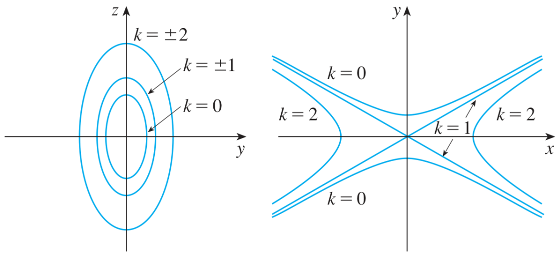


29–30 Sketch and identify a quadric surface that could have the traces shown.

29. Traces in $x = k$ Traces in $y = k$



30. Traces in $x = k$ Traces in $z = k$



31–38 Reduce the equation to one of the standard forms, classify the surface, and sketch it.

31. $y^2 = x^2 + \frac{1}{9}z^2$

32. $4x^2 - y + 2z^2 = 0$

33. $x^2 + 2y - 2z^2 = 0$

34. $y^2 = x^2 + 4z^2 + 4$

35. $x^2 + y^2 - 2x - 6y - z + 10 = 0$

36. $x^2 - y^2 - z^2 - 4x - 2z + 3 = 0$

37. $x^2 - y^2 + z^2 - 4x - 2z = 0$

38. $4x^2 + y^2 + z^2 - 24x - 8y + 4z + 55 = 0$

39–42 Use a computer with three-dimensional graphing software to graph the surface. Experiment with viewpoints and with domains for the variables until you get a good view of the surface.

39. $-4x^2 - y^2 + z^2 = 1$

40. $x^2 - y^2 - z = 0$

41. $-4x^2 - y^2 + z^2 = 0$

42. $x^2 - 6x + 4y^2 - z = 0$

43. Sketch the region bounded by the surfaces $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 = 1$ for $1 \leqslant z \leqslant 2$.

44. Sketch the region bounded by the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

45. Find an equation for the surface obtained by rotating the curve $y = \sqrt{x}$ about the x-axis.

46. Find an equation for the surface obtained by rotating the line $z = 2y$ about the z-axis.

47. Find an equation for the surface consisting of all points that are equidistant from the point $(-1, 0, 0)$ and the plane $x = 1$. Identify the surface.

48. Find an equation for the surface consisting of all points P for which the distance from P to the x-axis is twice the distance from P to the yz-plane. Identify the surface.

49. Traditionally, the earth's surface has been modeled as a sphere, but the World Geodetic System of 1984 (WGS-84) uses an ellipsoid as a more accurate model. It places the center of the earth at the origin and the north pole on the positive z-axis. The distance from the center to the poles is 6356.523 km and the distance to a point on the equator is 6378.137 km.
(a) Find an equation of the earth's surface as used by WGS-84.
(b) Curves of equal latitude are traces in the planes $z = k$. What is the shape of these curves?
(c) Meridians (curves of equal longitude) are traces in planes of the form $y = mx$. What is the shape of these meridians?

50. A cooling tower for a nuclear reactor is to be constructed in the shape of a hyperboloid of one sheet (see the photo on page 879). The diameter at the base is 280 m and the minimum diameter, 500 m above the base, is 200 m. Find an equation for the tower.

51. Show that if the point (a, b, c) lies on the hyperbolic paraboloid $z = y^2 - x^2$, then the lines with parametric equations $x = a + t$, $y = b + t$, $z = c + 2(b - a)t$ and $x = a + t$, $y = b - t$, $z = c - 2(b + a)t$ both lie entirely on this paraboloid. (This shows that the hyperbolic paraboloid is what is called a **ruled surface**; that is, it can be generated by the motion of a straight line. In fact, this exercise shows that through each point on the hyperbolic paraboloid there are two generating lines. The only other quadric surfaces that are ruled surfaces are cylinders, cones, and hyperboloids of one sheet.)

52. Show that the curve of intersection of the surfaces $x^2 + 2y^2 - z^2 + 3x = 1$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$ lies in a plane.

53. Graph the surfaces $z = x^2 + y^2$ and $z = 1 - y^2$ on a common screen using the domain $|x| \leq 1.2$, $|y| \leq 1.2$ and observe the curve of intersection of these surfaces. Show that the projection of this curve onto the xy-plane is an ellipse.

12 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. What is the difference between a vector and a scalar?

2. How do you add two vectors geometrically? How do you add them algebraically?

3. If **a** is a vector and c is a scalar, how is c**a** related to **a** geometrically? How do you find c**a** algebraically?

4. How do you find the vector from one point to another?

5. How do you find the dot product **a** · **b** of two vectors if you know their lengths and the angle between them? What if you know their components?

6. How are dot products useful?

7. Write expressions for the scalar and vector projections of **b** onto **a**. Illustrate with diagrams.

8. How do you find the cross product **a** × **b** of two vectors if you know their lengths and the angle between them? What if you know their components?

9. How are cross products useful?

10. (a) How do you find the area of the parallelogram determined by **a** and **b**?
(b) How do you find the volume of the parallelepiped determined by **a**, **b**, and **c**?

11. How do you find a vector perpendicular to a plane?

12. How do you find the angle between two intersecting planes?

13. Write a vector equation, parametric equations, and symmetric equations for a line.

14. Write a vector equation and a scalar equation for a plane.

15. (a) How do you tell if two vectors are parallel?
(b) How do you tell if two vectors are perpendicular?
(c) How do you tell if two planes are parallel?

16. (a) Describe a method for determining whether three points P, Q, and R lie on the same line.
(b) Describe a method for determining whether four points P, Q, R, and S lie in the same plane.

17. (a) How do you find the distance from a point to a line?
(b) How do you find the distance from a point to a plane?
(c) How do you find the distance between two lines?

18. What are the traces of a surface? How do you find them?

19. Write equations in standard form of the six types of quadric surfaces.

The **differential** dw is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz$$

EXAMPLE 6 The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

SOLUTION If the dimensions of the box are x, y, and z, its volume is $V = xyz$ and so

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz = yz\,dx + xz\,dy + xy\,dz$$

We are given that $|\Delta x| \leqslant 0.2$, $|\Delta y| \leqslant 0.2$, and $|\Delta z| \leqslant 0.2$. To estimate the largest error in the volume, we therefore use $dx = 0.2$, $dy = 0.2$, and $dz = 0.2$ together with $x = 75$, $y = 60$, and $z = 40$:

$$\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980$$

Thus an error of only 0.2 cm in measuring each dimension could lead to an error of approximately 1980 cm^3 in the calculated volume! This may seem like a large error, but it's only about 1% of the volume of the box. ■

14.4 EXERCISES

1–6 Find an equation of the tangent plane to the given surface at the specified point.

1. $z = 2x^2 + y^2 - 5y, \quad (1, 2, -4)$

2. $z = (x + 2)^2 - 2(y - 1)^2 - 5, \quad (2, 3, 3)$

3. $z = e^{x-y}, \quad (2, 2, 1)$

4. $z = x/y^2, \quad (-4, 2, -1)$

5. $z = x\sin(x + y), \quad (-1, 1, 0)$

6. $z = \ln(x - 2y), \quad (3, 1, 0)$

7–8 Graph the surface and the tangent plane at the given point. (Choose the domain and viewpoint so that you get a good view of both the surface and the tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

7. $z = x^2 + xy + 3y^2, \quad (1, 1, 5)$

8. $z = \sqrt{9 + x^2y^2}, \quad (2, 2, 5)$

CAS **9–10** Draw the graph of f and its tangent plane at the given point. (Use your computer algebra system both to compute the partial derivatives and to graph the surface and its tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

9. $f(x, y) = \dfrac{1 + \cos^2(x - y)}{1 + \cos^2(x + y)}, \quad \left(\dfrac{\pi}{3}, \dfrac{\pi}{6}, \dfrac{7}{4}\right)$

10. $f(x, y) = e^{-xy/10}\left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right), \quad (1, 1, 3e^{-0.1})$

11–16 Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

11. $f(x, y) = 1 + x\ln(xy - 5), \quad (2, 3)$

12. $f(x, y) = \sqrt{xy}, \quad (1, 4)$

13. $f(x, y) = x^2e^y, \quad (1, 0)$

14. $f(x, y) = \dfrac{1 + y}{1 + x}, \quad (1, 3)$

15. $f(x, y) = 4\arctan(xy), \quad (1, 1)$

16. $f(x, y) = y + \sin(x/y), \quad (0, 3)$

17–18 Verify the linear approximation at (0, 0).

17. $e^x\cos(xy) \approx x + 1$

18. $\dfrac{y - 1}{x + 1} \approx x + y - 1$

19. Given that f is a differentiable function with $f(2, 5) = 6$, $f_x(2, 5) = 1$, and $f_y(2, 5) = -1$, use a linear approximation to estimate $f(2.2, 4.9)$.

20. Find the linear approximation of the function $f(x, y) = 1 - xy \cos \pi y$ at $(1, 1)$ and use it to approximate $f(1.02, 0.97)$. Illustrate by graphing f and the tangent plane.

21. Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(3, 2, 6)$ and use it to approximate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.

22. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table. Use the table to find a linear approximation to the wave height function when v is near 40 knots and t is near 20 hours. Then estimate the wave heights when the wind has been blowing for 24 hours at 43 knots.

Duration (hours)

Wind speed (knots): v \ t	5	10	15	20	30	40	50
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

23. Use the table in Example 3 to find a linear approximation to the heat index function when the temperature is near 94°F and the relative humidity is near 80%. Then estimate the heat index when the temperature is 95°F and the relative humidity is 78%.

24. The wind-chill index W is the perceived temperature when the actual temperature is T and the wind speed is v, so we can write $W = f(T, v)$. The following table of values is an excerpt from Table 1 in Section 14.1. Use the table to find a linear approximation to the wind-chill index function when T is near -15°C and v is near 50 km/h. Then estimate the wind-chill index when the temperature is -17°C and the wind speed is 55 km/h.

Wind speed (km/h)

Actual temperature (°C): T \ v	20	30	40	50	60	70
−10	−18	−20	−21	−22	−23	−23
−15	−24	−26	−27	−29	−30	−30
−20	−30	−33	−34	−35	−36	−37
−25	−37	−39	−41	−42	−43	−44

25–30 Find the differential of the function.

25. $z = e^{-2x} \cos 2\pi t$

26. $u = \sqrt{x^2 + 3y^2}$

27. $m = p^5 q^3$

28. $T = \dfrac{v}{1 + uvw}$

29. $R = \alpha\beta^2 \cos \gamma$

30. $L = xze^{-y^2 - z^2}$

31. If $z = 5x^2 + y^2$ and (x, y) changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of Δz and dz.

32. If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of Δz and dz.

33. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

34. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

35. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

36. The wind-chill index is modeled by the function

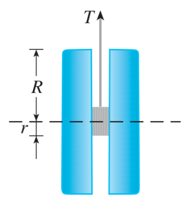
$$W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

where T is the temperature (in °C) and v is the wind speed (in km/h). The wind speed is measured as 26 km/h, with a possible error of ± 2 km/h, and the temperature is measured as -11°C, with a possible error of ± 1°C. Use differentials to estimate the maximum error in the calculated value of W due to the measurement errors in T and v.

37. The tension T in the string of the yo-yo in the figure is

$$T = \frac{mgR}{2r^2 + R^2}$$

where m is the mass of the yo-yo and g is acceleration due to gravity. Use differentials to estimate the change in the tension if R is increased from 3 cm to 3.1 cm and r is increased from 0.7 cm to 0.8 cm. Does the tension increase or decrease?



38. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

39. If R is the total resistance of three resistors, connected in parallel, with resistances R_1, R_2, R_3, then

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If the resistances are measured in ohms as $R_1 = 25\ \Omega$, $R_2 = 40\ \Omega$, and $R_3 = 50\ \Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of R.

40. A model for the surface area of a human body is given by $S = 0.1091w^{0.425}h^{0.725}$, where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet. If the errors in measurement of w and h are at most 2%, use differentials to estimate the maximum percentage error in the calculated surface area.

41. In Exercise 14.1.39 and Example 14.3.3, the body mass index of a person was defined as $B(m, h) = m/h^2$, where m is the mass in kilograms and h is the height in meters.

(a) What is the linear approximation of $B(m, h)$ for a child with mass 23 kg and height 1.10 m?

(b) If the child's mass increases by 1 kg and height by 3 cm, use the linear approximation to estimate the new BMI. Compare with the actual new BMI.

42. Suppose you need to know an equation of the tangent plane to a surface S at the point $P(2, 1, 3)$. You don't have an equation for S but you know that the curves

$$\mathbf{r}_1(t) = \langle 2 + 3t,\ 1 - t^2,\ 3 - 4t + t^2 \rangle$$
$$\mathbf{r}_2(u) = \langle 1 + u^2,\ 2u^3 - 1,\ 2u + 1 \rangle$$

both lie on S. Find an equation of the tangent plane at P.

43–44 Show that the function is differentiable by finding values of ε_1 and ε_2 that satisfy Definition 7.

43. $f(x, y) = x^2 + y^2$

44. $f(x, y) = xy - 5y^2$

45. Prove that if f is a function of two variables that is differentiable at (a, b), then f is continuous at (a, b).
Hint: Show that

$$\lim_{(\Delta x, \Delta y) \to (0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$$

46. (a) The function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

was graphed in Figure 4. Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but f is not differentiable at $(0, 0)$. [*Hint:* Use the result of Exercise 45.]

(b) Explain why f_x and f_y are not continuous at $(0, 0)$.

APPLIED PROJECT THE SPEEDO LZR RACER

Many technological advances have occurred in sports that have contributed to increased athletic performance. One of the best known is the introduction, in 2008, of the Speedo LZR racer. It was claimed that this full-body swimsuit reduced a swimmer's drag in the water. Figure 1 shows the number of world records broken in men's and women's long-course freestyle swimming events from 1990 to 2011.[1] The dramatic increase in 2008 when the suit was introduced led people to claim that such suits are a form of technological doping. As a result all full-body suits were banned from competition starting in 2010.

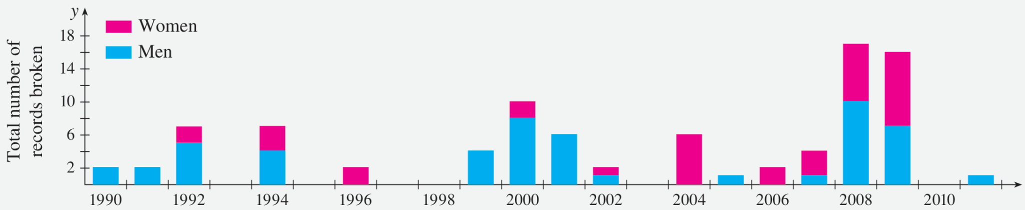


FIGURE 1 Number of world records set in long-course men's and women's freestyle swimming event 1990–2011

It might be surprising that a simple reduction in drag could have such a big effect on performance. We can gain some insight into this using a simple mathematical model.[2]

1. L. Foster et al., "Influence of Full Body Swimsuits on Competitive Performance," *Procedia Engineering* 34 (2012): 712–17.
2. Adapted from http://plus.maths.org/content/swimming.