15.2 EXERCISES

1–6 Evaluate the iterated integral.

1. $\int_1^5 \int_0^x (8x - 2y)\, dy\, dx$

2. $\int_0^2 \int_0^{y^2} x^2 y\, dx\, dy$

3. $\int_0^1 \int_0^y x e^{y^3}\, dx\, dy$

4. $\int_0^{\pi/2} \int_0^x x \sin y\, dy\, dx$

5. $\int_0^1 \int_0^{s^2} \cos(s^3)\, dt\, ds$

6. $\int_0^1 \int_0^{e^v} \sqrt{1 + e^v}\, dw\, dv$

7–10 Evaluate the double integral.

7. $\iint_D \frac{y}{x^2 + 1}\, dA, \quad D = \{(x, y) \mid 0 \leqslant x \leqslant 4, 0 \leqslant y \leqslant \sqrt{x}\}$

8. $\iint_D (2x + y)\, dA, \quad D = \{(x, y) \mid 1 \leqslant y \leqslant 2, y - 1 \leqslant x \leqslant 1\}$

9. $\iint_D e^{-y^2}\, dA, \quad D = \{(x, y) \mid 0 \leqslant y \leqslant 3, 0 \leqslant x \leqslant y\}$

10. $\iint_D y\sqrt{x^2 - y^2}\, dA, \quad D = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant x\}$

11. Draw an example of a region that is
(a) type I but not type II
(b) type II but not type I

12. Draw an example of a region that is
(a) both type I and type II
(b) neither type I nor type II

13–14 Express D as a region of type I and also as a region of type II. Then evaluate the double integral in two ways.

13. $\iint_D x\, dA, \quad D$ is enclosed by the lines $y = x, y = 0, x = 1$

14. $\iint_D xy\, dA, \quad D$ is enclosed by the curves $y = x^2, y = 3x$

15–16 Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

15. $\iint_D y\, dA, \quad D$ is bounded by $y = x - 2, x = y^2$

16. $\iint_D y^2 e^{xy}\, dA, \quad D$ is bounded by $y = x, y = 4, x = 0$

17–22 Evaluate the double integral.

17. $\iint_D x \cos y\, dA, \quad D$ is bounded by $y = 0, \ y = x^2, \ x = 1$

18. $\iint_D (x^2 + 2y)\, dA, \quad D$ is bounded by $y = x, y = x^3, x \geqslant 0$

19. $\iint_D y^2\, dA,$
D is the triangular region with vertices (0, 1), (1, 2), (4, 1)

20. $\iint_D xy\, dA, \quad D$ is enclosed by the quarter-circle
$y = \sqrt{1 - x^2}, x \geqslant 0$, and the axes

21. $\iint_D (2x - y)\, dA,$
D is bounded by the circle with center the origin and radius 2

22. $\iint_D y\, dA, \quad D$ is the triangular region with vertices (0, 0), (1, 1), and (4, 0)

23–32 Find the volume of the given solid.

23. Under the plane $3x + 2y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$

24. Under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$

25. Under the surface $z = xy$ and above the triangle with vertices (1, 1), (4, 1), and (1, 2)

26. Enclosed by the paraboloid $z = x^2 + y^2 + 1$ and the planes $x = 0, y = 0, z = 0$, and $x + y = 2$

27. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

28. Bounded by the planes $z = x, y = x, x + y = 2$, and $z = 0$

29. Enclosed by the cylinders $z = x^2, y = x^2$ and the planes $z = 0, y = 4$

30. Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0, z = 0$ in the first octant

31. Bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0, z = 0$ in the first octant

32. Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$

33. Use a graphing calculator or computer to estimate the x-coordinates of the points of intersection of the curves $y = x^4$ and $y = 3x - x^2$. If D is the region bounded by these curves, estimate $\iint_D x\, dA$.

34. Find the approximate volume of the solid in the first octant that is bounded by the planes $y = x$, $z = 0$, and $z = x$ and the cylinder $y = \cos x$. (Use a graphing device to estimate the points of intersection.)

35–38 Find the volume of the solid by subtracting two volumes.

35. The solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$, $2x + 2y - z + 10 = 0$

36. The solid enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$

37. The solid under the plane $z = 3$, above the plane $z = y$, and between the parabolic cylinders $y = x^2$ and $y = 1 - x^2$

38. The solid in the first octant under the plane $z = x + y$, above the surface $z = xy$, and enclosed by the surfaces $x = 0$, $y = 0$, and $x^2 + y^2 = 4$

39–40 Sketch the solid whose volume is given by the iterated integral.

39. $\int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$

40. $\int_0^1 \int_0^{1-x^2} (1 - x)\, dy\, dx$

CAS **41–44** Use a computer algebra system to find the exact volume of the solid.

41. Under the surface $z = x^3y^4 + xy^2$ and above the region bounded by the curves $y = x^3 - x$ and $y = x^2 + x$ for $x \geqslant 0$

42. Between the paraboloids $z = 2x^2 + y^2$ and $z = 8 - x^2 - 2y^2$ and inside the cylinder $x^2 + y^2 = 1$

43. Enclosed by $z = 1 - x^2 - y^2$ and $z = 0$

44. Enclosed by $z = x^2 + y^2$ and $z = 2y$

45–50 Sketch the region of integration and change the order of integration.

45. $\int_0^1 \int_0^y f(x, y)\, dx\, dy$

46. $\int_0^2 \int_{x^2}^4 f(x, y)\, dy\, dx$

47. $\int_0^{\pi/2} \int_0^{\cos x} f(x, y)\, dy\, dx$

48. $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} f(x, y)\, dx\, dy$

49. $\int_1^2 \int_0^{\ln x} f(x, y)\, dy\, dx$

50. $\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y)\, dy\, dx$

51–56 Evaluate the integral by reversing the order of integration.

51. $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$

52. $\int_0^1 \int_{x^2}^1 \sqrt{y} \sin y\, dy\, dx$

53. $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1}\, dy\, dx$

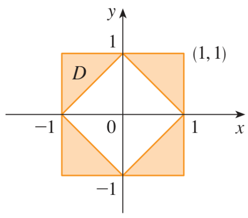
54. $\int_0^2 \int_{y/2}^1 y \cos(x^3 - 1)\, dx\, dy$

55. $\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, dx\, dy$

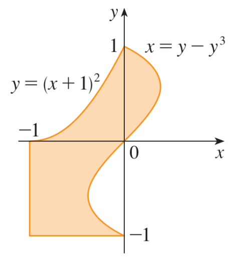
56. $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$

57–58 Express D as a union of regions of type I or type II and evaluate the integral.

57. $\iint_D x^2\, dA$



58. $\iint_D y\, dA$



59–60 Use Property 11 to estimate the value of the integral.

59. $\iint_S \sqrt{4 - x^2y^2}\, dA$, $S = \{(x, y) \mid x^2 + y^2 \leqslant 1, x \geqslant 0\}$

60. $\iint_T \sin^4(x + y)\, dA$, T is the triangle enclosed by the lines $y = 0$, $y = 2x$, and $x = 1$

61–62 Find the averge value of f over the region D.

61. $f(x, y) = xy$, D is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$

62. $f(x, y) = x \sin y$, D is enclosed by the curves $y = 0$, $y = x^2$, and $x = 1$

63. Prove Property 11.

64. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint_D f(x, y)\, dA = \int_0^1 \int_0^{2y} f(x, y)\, dx\, dy + \int_1^3 \int_0^{3-y} f(x, y)\, dx\, dy$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.

65–69 Use geometry or symmetry, or both, to evaluate the double integral.

65. $\displaystyle\iint_D (x + 2)\, dA,$

$D = \{(x, y) \mid 0 \leqslant y \leqslant \sqrt{9 - x^2}\}$

66. $\displaystyle\iint_D \sqrt{R^2 - x^2 - y^2}\, dA,$

D is the disk with center the origin and radius R

67. $\displaystyle\iint_D (2x + 3y)\, dA,$

D is the rectangle $0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b$

68. $\displaystyle\iint_D (2 + x^2y^3 - y^2 \sin x)\, dA,$

$D = \{(x, y) \mid |x| + |y| \leqslant 1\}$

69. $\displaystyle\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right) dA,$

$D = [-a, a] \times [-b, b]$

CAS **70.** Graph the solid bounded by the plane $x + y + z = 1$ and the paraboloid $z = 4 - x^2 - y^2$ and find its exact volume. (Use your CAS to do the graphing, to find the equations of the boundary curves of the region of integration, and to evaluate the double integral.)

15.3 Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\iint_R f(x, y)\, dA$, where R is one of the regions shown in Figure 1. In either case the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates.

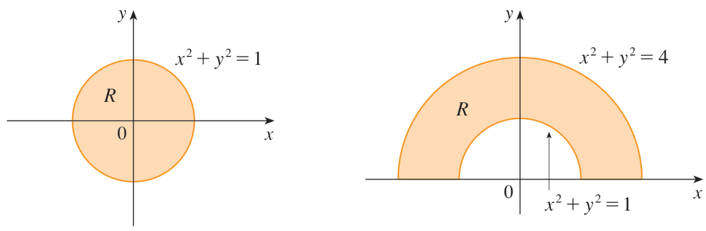


FIGURE 1

(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$

(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

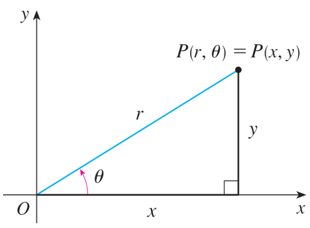


FIGURE 2

Recall from Figure 2 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

(See Section 10.3.)

The regions in Figure 1 are special cases of a **polar rectangle**

$$R = \{(r, \theta) \mid a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta\}$$

which is shown in Figure 3. In order to compute the double integral $\iint_R f(x, y)\, dA$, where R is a polar rectangle, we divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b - a)/m$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = (\beta - \alpha)/n$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles R_{ij} shown in Figure 4.